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# Technical Notes

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## Complete Radiative Terms for the Electron/Electronic Energy Equation

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### Nomenclature

$a_{i,n}$ = stoichiometric coefficient, –
$B_{e^-,\mathrm{elct}}$ = energy transfer between free electrons and electronic states, W/cm$^3$
$D$ = dissociation energy, eV
$E$ = energy per particle, eV
$e$ = energy per unit mass, erg/g
$e''$ = electron/electronic energy, erg/g
$h\nu$ = photon energy, eV
$I$ = ionization energy, eV
$k$ = absorption coefficient, 1/cm
$m$ = mass of a particle, g
$n_{\mathrm{diss}}$ = number of dissociation processes, –
$n_{\mathrm{ion}}$ = number of ionization processes, –
$n_{M\text{-band}}$ = number of molecular bands, –
$n_s$ = number of species, –
$p$ = pressure, dyne/cm$^2$
$Q$ = energy production due to inelastic collisions, W/cm$^3$
$\bar{q}$ = radiative flux vector, W/cm$^2$
$V$ = velocity magnitude, cm/s
$\bar{V}$ = velocity vector, cm/s
$\dot{w}$ = mass production rate, g/cm$^3$ s
$X_n$ = chemical symbol for the $n$th species, –
$\bar{x}$ = direction vector, cm
$Y$ = absorption coefficient ratio, –
$\nu$ = frequency, 1/s
$\xi$ = energy production due to elastic collisions, W/cm$^3$
$\rho$ = density, g/cm$^3$

*Subscripts*
elct = electronic
$e^-$ = free electron
$e^0$ = zero point
$i$ = for the $i$th radiative process
$n$ = for the $n$th species
rot = rotational
tr = translational
vib = vibrational
$\nu$ = at the frequency $\nu$

*Superscripts*
$a1$, $a2$ = atoms $a1$ and $a2$
coll = due to collisional processes
$l$ = lower energy state
$m$ = molecule
rad = due to radiative processes
tot = for all radiative processes
$u$ = upper energy state

### Introduction

A COMMON method of including the radiative transfer effects in the flowfield calculations of a multitemperature gas is to assume that the total quantity of radiative energy absorbed or emitted by the gas affects only the electron and electronic energies.[1–3] However, the energy mode of the gas affected by the absorption and emission of radiation depends on the type of radiative process involved, as well as the frequency of the photon absorbed or emitted. Additionally, as radiative chemistry becomes more important in the flowfield, a significant portion of the energy absorbed and emitted is associated with changes in the zero point energy of the gas. The greater the effects of radiative chemistry become, the poorer this common method actually models the physics of the gas.

The purpose of this Note is to derive the radiative terms in the electron/electronic energy equation, properly accounting for the effects of absorption and emission of radiation on the individual energy modes of the gas. This electron/electronic energy equation with the complete radiative terms has successfully been used to model the radiation-dominated precursor ahead of the bow shock of a hypersonic vehicle entering the Earth's atmosphere.[4]

### Derivation of Radiative Terms

The total rate of change in the energy of a gas due to the absorption and emission of radiation is equal to the negative of the divergence of the radiative flux vector, $-(\partial/\partial\bar{x})\cdot\bar{q}$, since the energy lost by the gas due to emission is transferred into radiative energy and appears as an increase in the radiative flux. This change can be split into components, one for each energy mode of the gas

$$\frac{\partial}{\partial\bar{x}}\cdot\bar{q} = \left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)_{e^-} + \left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)_{\mathrm{tr}} + \left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)_{\mathrm{elct}} + \left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)_{\mathrm{vib}} + \left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)_{\mathrm{rot}} + \left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)_{e^0} \tag{1}$$

The terms on the right side of this equation represent the six possible modes of energy storage of the gas.

For this derivation, we will utilize the free electron energy equation derived by Carlson[5]

$$\frac{\partial}{\partial t}(\rho_{e^-}e_{e^-}) + \frac{\partial}{\partial\bar{x}}\cdot(\rho_{e^-}e_{e^-}\bar{V}) = -p_{e^-}\frac{\partial}{\partial\bar{x}}\cdot\bar{V} + \sum_{n=1}^{n_s}\xi_{e^-,n} + Q_{e^-} + \dot{w}_{e^-}\frac{V^2}{2} - \left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)_{e^-} + B_{e^-,\mathrm{elct}} \tag{2}$$

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Carlson's derivation of this equation was based on the detailed approach of Chapman and Cowling[6] and begins with their general equation of change. In this equation, the terms involving magnetic and electric fields have been neglected. The second and third terms on the right side of this equation contain the effects of all elastic and inelastic collisions, respectively. The specific form of these terms is not presented here since the emphasis of this Note is on the radiative terms. Likewise, since the diffusion, heat transfer, and viscous terms are not affected by the inclusion of the radiative transfer terms, they have been neglected for simplicity. This general form of the electron energy equation has been used in calculations of the viscous shock layer[7] as well as the shock wave precursor.[4]

The last two terms on the right side of Eq. (2) have been added to Carlson's equation. The first of these accounts for the effects of absorption and emission of radiation, whereas the last term accounts for the transfer of energy between the free electron and the electronic states. It should be noted that the radiation term included above is not the total change in the radiative flux. This term includes only the portion of the radiation that affects the free electron energy. Also, if it is assumed that the free electrons and electronic states are in equilibrium, the last term must be included to maintain this equilibrium.

Equation (2) is written in the form of a conservation of free electron energy per unit volume. The equivalent equation for the electronic energy of the gas can be written as

$$\frac{\partial}{\partial t}\left(\sum_{n=1}^{n_s} \rho_n e_{\mathrm{elct}_n}\right) + \frac{\partial}{\partial \bar{x}}\cdot\left(\bar{V}\sum_{n=1}^{n_s} \rho_n e_{\mathrm{elct}_n}\right) = -\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)_{\mathrm{elct}} + Q_{\mathrm{elct}} - B_{e^-,\mathrm{elct}} \tag{3}$$

The first term on the right side of this equation accounts for the effect of radiation absorption on the electronic energy. The second term accounts for the production of electronic energy through inelastic collisions and the subsequent production of species having electronic energy. As discussed above, the form of the collisional term is not important to this derivation, and is therefore not presented.

If it assumed that the free electrons and the electronic energy modes are in equilibrium, thus governed by the same temperature, it is appropriate to use a combination electron/electronic energy equation. This approach is also advantageous since it eliminates the energy transfer term between the electrons and electronic states, which is neither well defined nor well understood. Also, since a large portion of the radiative energy absorbed affects the energy stored in the zero points of the gas, it is convenient to add these terms into the electron/electronic energy equation. Adding these terms helps to facilitate the use of the entire change in the radiative flux and will not have any adverse effect on the equation since they do not directly depend on temperature. Summing the equation for conservation of mass for each species multiplied by the zero point energy of the species yields

$$\frac{\partial}{\partial t}\left(\sum_{n=1}^{n_s} \rho_n e_{e_n^0}\right) + \frac{\partial}{\partial \bar{x}}\cdot\left(\bar{V}\sum_{n=1}^{n_s} \rho_n e_{e_n^0}\right) = \sum_{n=1}^{n_s} \dot{w}_n e_{e_n^0} = \sum_{n=1}^{n_s} (\dot{w}_n^{\mathrm{coll}} + \dot{w}_n^{\mathrm{rad}}) e_{e_n^0} \tag{4}$$

This equation represents the conservation of zero point energy of the gas. It should be noted that the mass production rates on the right side of this equation include the effects of both collisional $\dot{w}_n^{\mathrm{coll}}$ and radiative $\dot{w}_n^{\mathrm{rad}}$ reactions. Adding Eqs. (2), (3), and (4), and incorporating Eq. (1) for the radiative terms leads to

$$\frac{\partial}{\partial t}(\rho e'') + \frac{\partial}{\partial \bar{x}}\cdot(\bar{V}\rho e'') = -p_{e^-}\frac{\partial}{\partial \bar{x}}\cdot\bar{V} + \sum_{n=1}^{n_s} \xi_{e^-,n} + Q_{e^-} + Q_{\mathrm{elct}} + \dot{w}_{e^-}\frac{V^2}{2} + \sum_{n=1}^{n_s} \dot{w}_n^{\mathrm{coll}} e_{e_n^0} + \sum_{n=1}^{n_s} \dot{w}_n^{\mathrm{rad}} e_{e_n^0} - \frac{\partial}{\partial \bar{x}}\cdot\bar{q} + \left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)_{\mathrm{vib}} + \left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)_{\mathrm{tr}} + \left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)_{\mathrm{rot}} + \left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)_{e^0} \tag{5}$$

where

$$e'' = \frac{\rho_{e^-}}{\rho} e_{e^-} + \sum_{n=1}^{n_s} \frac{\rho_n}{\rho}(e_{\mathrm{elct}_n} + e_{e_n^0}) \tag{6}$$

This equation represents the conservation of electron, electronic, and zero point energy of the gas. The last six terms on the right side of this equation include the effects of the absorption and emission of radiation. At first it would appear that the last term in this equation should not be included; however, it will be seen that this term cancels with the radiation mass production term effectively removing them both from the equation. The remainder of this Note will be spent deriving expressions for the last four terms in this equation.

Defining the absorption coefficient ratio as $Y_{\nu_i} = k_{\nu_i}/k_\nu^{\mathrm{tot}}$ under the quasiequilibrium hypothesis[8] it is possible to find the divergence of the radiative flux due to the $i$th process

$$\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)_i = \int_0^\infty Y_{\nu_i}\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \tag{7}$$

Likewise, since the energy of a photon is $h\nu$, the rate of the photon absorption per unit volume for the process can be written as

$$\left(\frac{\text{rate of photons absorbed}}{\text{volume}}\right)_i = \int_0^\infty \frac{Y_{\nu_i}}{h\nu}\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \tag{8}$$

Using an absorption coefficient ratio in these calculations is a matter of convenience. It allows the radiation calculations to be performed using the total absorption coefficient and with a small amount of bookkeeping the effects of each individual radiative process can be separated for later use in the flowfield calculations. This use of an absorption coefficient ratio is a new idea, however, these values are easily calculated from the individual process absorption coefficients used to determine the total absorption coefficient in the radiation calculations.

Since each radiative process affects the individual energy modes of the gas differently, it is important to separate the total absorption at each wavelength into that due to each type of process. Some radiative processes, such as free-free emission and absorption, affect only the free electron kinetic energy, while others, photodissociation processes, affect all of the other energy modes of the gas while having no effect on the free electron kinetic energy.

At this point, it is necessary to consider each radiative process and quantify its effects on the individual energy modes of the gas. The absorption and emission of radiation through the free-free processes affects only the kinetic energy of the free electrons. Using Eq. (7), we have

$$\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)_{e^-}^{\mathrm{FF}} = \int_0^\infty Y_\nu^{\mathrm{FF}}\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \tag{9}$$

Likewise, since atomic line radiation involves only transitions between the electronic states of an atom, the radiation emitted or absorbed through these processes influences only the electronic energy. As with the free-free processes above, this affect can be determined with the use of Eq. (7)

$$\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)^{A\text{-lines}}_{\mathrm{elct}} = \int_0^\infty Y_\nu^{A\text{-lines}}\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \tag{10}$$

In this equation, the variable $Y_\nu^{A\text{-lines}}$ includes the absorption coefficients for all of the atomic lines.

The absorption and emission of radiation through molecular bands involves not only the electronic states, but also the vibrational and rotational states. The change in the electronic energy of a particle due to the absorption of a single photon through a molecular band is, $\Delta E_{\mathrm{elct}} = E^u_{\mathrm{elct}} - E^l_{\mathrm{elct}}$. Using this and Eq. (8), the total change in the electronic energy due to molecular bands is given by

$$\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)^{M\text{-band}}_{\mathrm{elct}} = \sum_{i=1}^{n_{M\text{-band}}}\int_0^\infty \frac{Y_{\nu_i}(E^u_{\mathrm{elct}_i} - E^l_{\mathrm{elct}_i})}{h\nu}\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \tag{11}$$

The integration in this equation accounts for all photons absorbed or emitted through a single molecular band, whereas the summation is over all molecular band processes.

The effect of absorption through the molecular bands on the vibrational and rotational energies of a gas is not easily calculated without exact knowledge of the upper and lower energy levels of the molecule involved. However, the sum of the effect on these two energy modes can be found. The absorption of a photon through a molecular band affects the electronic, vibrational, and rotational energies of the gas. An energy balance for the absorption of a single photon, therefore, must include terms for all of these energy modes, $h\nu = \Delta E_{\mathrm{elct}} + \Delta E_{\mathrm{vib}} + \Delta E_{\mathrm{rot}}$. This expression along with Eq. (8) leads to

$$\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)^{M\text{-band}}_{\mathrm{vib}} + \left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)^{M\text{-band}}_{\mathrm{rot}}$$

$$= \sum_{i=1}^{n_{M\text{-band}}}\int_0^\infty \frac{Y_{\nu_i} - (h\nu - E^u_{\mathrm{elct}_i} + E^l_{\mathrm{elct}_i})}{h\nu}\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \tag{12}$$

for the sum of the change in the vibrational and rotational energies due to the absorption and emission of radiation through the molecular bands.

The effects of the bound-free absorption processes on the zero point energy of the gas can be considered as a whole rather than separating into photoionization and photodissociation. This is appropriate since all of the chemical reactions related to the bound-free photoprocesses can be represented by

$$\sum_{n=1}^{n_s} a_{i,n}X_n + h\nu \underset{}{\overset{k_{\nu_i}}{\rightleftharpoons}} \sum_{n=1}^{n_s} a'_{i,n}X_n \tag{13}$$

and the energy necessary for this reaction to occur is simply the difference between the zero point energies of the products and the reactants. For the ionization reaction this difference is equivalent to $I$, and for the dissociation reaction it is $D$. For each photon absorbed through the $i$th photoprocess, this reaction energy is given by

$$I_i,\ D_i = \sum_{n=1}^{n_s}(a'_{i,n} - a_{i,n})m_n e_{e_n^0} \tag{14}$$

Using Eqs. (8) and (14), the change in the zero point energy of the gas due to absorption or emission of radiation in the bound-free processes, both ionization and dissociation, is given by

$$\Delta e_{e_i^0} = \int_0^\infty \frac{Y_{\nu_i}}{h\nu}\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)\sum_{n=1}^{n_s}[(a'_{i,n} - a_{i,n})m_n e_{e_i^0}]\,\mathrm{d}\nu \tag{15}$$

Considering the photoreaction given by Eq. (13), if every photon absorbed by the associated photoprocess causes the reaction to occur, then using Eq. (8) the mass production of the $n$th species due to the $i$th reaction is given by

$$\dot{w}_{n_i} = \int_0^\infty \frac{Y_{\nu_i}}{h\nu}m_n(a'_{i,n} - a_{i,n})\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \tag{16}$$

Substituting this equation for the mass production of each species into Eq. (15) and summing $i$ over all radiative reactions leads to the expression

$$\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)^{\mathrm{ion}}_{e^0} + \left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)^{\mathrm{diss}}_{e^0} = \sum_{n=1}^{n_s}\dot{w}_n^{\mathrm{rad}}e_{e_n^0} \tag{17}$$

for the change in the zero point energy of the gas due to photoionization and photodissociation.

The influence of the bound-free processes on the remaining energy modes will be found by first considering the ionization process and then the dissociation process. The ionization process affects the electronic, free electron, and zero point energies of the gas. For each photon absorbed, the electronic energy of the gas decreases by the electronic energy $E_{\mathrm{elct}}$ of the particle absorbing the photon. This assumes that the ion formed in the process is in the ground state and thus has no electronic energy. Using Eq. (8) and summing over the photoionization processes, the change in the electronic energy due to these reactions is

$$\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)^{\mathrm{ion}}_{\mathrm{elct}} = \sum_{i=1}^{n_{\mathrm{ion}}}\int_0^\infty \frac{Y_{\nu_i}(-E_{\mathrm{elct}_i})}{h\nu}\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \tag{18}$$

Due to the photoionization process, the energy of the free electrons increases by the kinetic energy of the electron released. Using the energy balance for a photoionization process, $h\nu + E_{\mathrm{elct}} = I + E_{e^-}$, and Eq. (8), the effect of photoionization on the free electron energy is given by

$$\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)^{\mathrm{ion}}_{e^-} = \sum_{i=1}^{n_{\mathrm{ion}}}\int_0^\infty \frac{Y_{\nu_i}(h\nu + E_{\mathrm{elct}_i} - I_i)}{h\nu}\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \tag{19}$$

where the summation is over all photoionization processes. This expression assumes that the ion formed through photoionization of a polyatomic molecule has the same vibrational and rotational energy as the parent molecule.

The final process to be considered is that of photodissociation. This radiative process affects the electronic, vibrational, rotational, translational, and zero point energies of the gas. The change in the electronic energy can be determined from the knowledge of the molecular band through which dissociation occurs. The change in the electronic energy due to a single photodissociation is $\Delta E_{\mathrm{elct}} = E^{a1}_{\mathrm{elct}} + E^{a2}_{\mathrm{elct}} - E^m_{\mathrm{elct}}$. In this equation, the energy of the electronic state for each particle is considered relative to the ground state of the particle. Using this equation along with Eq. (8) and summing over the bound-free dissociation processes leads to the expression

$$\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}\right)^{\mathrm{diss}}_{\mathrm{elct}} = \sum_{i=1}^{n_{\mathrm{diss}}}\int_0^\infty \frac{Y_{\nu_i}(E^{a1}_{\mathrm{elct}_i} + E^{a2}_{\mathrm{elct}_i} - E^m_{\mathrm{elct}_i})}{h\nu}\left(\frac{\partial}{\partial \bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \tag{20}$$

for the change in the electronic energy due to photodissociation.

In a photodissociation process, the vibrational and rotational energy modes of gas lose energy equal to the vibrational energy $E^m_{\text{vib}}$ and the rotational energy $E^m_{\text{rot}}$ of the molecule dissociated. Therefore, using Eq. (8), the sum of the change in the vibrational and rotational energies of the gas due to the photodissociation process is

$$\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)^{\text{diss}}_{\text{vib}} + \left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)^{\text{diss}}_{\text{rot}} = \sum_{i=1}^{n_{\text{diss}}}\int_0^\infty \frac{Y_{\nu_i}(-E^m_{\text{vib}_i} - E^m_{\text{rot}_i})}{h\nu}\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \tag{21}$$

It should be noted that this assumes diatomic molecules so that the products of the dissociation have no vibrational or rotational energy.

Lastly, the translational energy of the heavy particles is changed by the difference in the translational energy of the two atoms released and that of the molecule consumed in the dissociation process. Utilizing the energy balance for a dissociation process

$$E^m_{\text{tr}} + E^m_{\text{elct}} + E^m_{\text{vib}} + E^m_{\text{rot}} + h\nu = D + E^{a1}_{\text{elct}} + E^{a2}_{\text{elct}} + E^{a1,a2}_{\text{tr}} \tag{22}$$

and the equation above for the change in the electronic energy, the change in the heavy particle translational energy due to the absorption of a photon in the $i$th dissociation continuum is given by

$$E^{a1,a2}_{\text{tr}_i} - E^m_{\text{tr}_i} = -\Delta E_{\text{elct}_i} + E^m_{\text{vib}_i} + E^m_{\text{vib}_i} + E^m_{\text{rot}_i} + h\nu - D_i \tag{23}$$

From this expression and Eq. (8), the total change in the heavy particle translational energy due to the photodissociation process is given by

$$\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)^{\text{diss}}_{\text{tr}} = \sum_{i=1}^{n_{\text{diss}}}\int_0^\infty \frac{Y_{\nu_i}(-\Delta E_{\text{elct}_i} + E^m_{\text{vib}_i} + E^m_{\text{rot}_i} + h\nu - D_i)}{h\nu} \times \left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \tag{24}$$

The total change in each energy mode due to photoprocesses is given by summing the contributions due to each type of photoprocess. This approach leads to the following relations:

Free electron kinetic

$$\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)_{e^-} = \int_0^\infty Y^{\text{FF}}_\nu\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu + \sum_{i=1}^{n_{\text{ion}}}\int_0^\infty \frac{Y_{\nu_i}(h\nu + E_{\text{elct}_i} - I_i)}{h\nu}\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \tag{25}$$

Heavy particle translational

$$\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)_{\text{tr}} = \sum_{i=1}^{n_{\text{diss}}}\int_0^\infty \frac{Y_{\nu_i}(-\Delta E_{\text{elct}_i} + E^m_{\text{vib}_i} + E^m_{\text{rot}_i} + h\nu - D_i)}{h\nu} \times \left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \tag{26}$$

Electronic

$$\begin{aligned}\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)_{\text{elct}} &= \int_0^\infty Y^{A\text{-Lines}}_\nu\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \\ &+ \sum_{i=1}^{n_{M\text{-band}}}\int_0^\infty \frac{Y_{\nu_i}(E^u_{\text{elct}_i} - E^l_{\text{elct}_i})}{h\nu}\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \\ &+ \sum_{i=1}^{n_{\text{ion}}}\int_0^\infty \frac{Y_{\nu_i}(-E_{\text{elct}_i})}{h\nu}\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \\ &+ \sum_{i=1}^{n_{\text{diss}}}\int_0^\infty \frac{Y_{\nu_i}(E^{a1}_{\text{elct}_i} + E^{a2}_{\text{elct}_i} - E^m_{\text{elct}_i})}{h\nu}\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu\end{aligned} \tag{27}$$

Vibrational and rotational

$$\begin{aligned}\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)_{\text{vib}} + \left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)_{\text{rot}} &= \sum_{i=1}^{n_{M\text{-band}}}\int_0^\infty \frac{Y_{\nu_i}(h\nu - E^u_{\text{elct}_i} + E^l_{\text{elct}_i})}{h\nu}\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \\ &+ \sum_{i=1}^{n_{\text{diss}}}\int_0^\infty \frac{Y_{\nu_i}(-E^m_{\text{vib}_i} - E^m_{\text{rot}_i})}{h\nu}\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu\end{aligned} \tag{28}$$

Zero point

$$\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}\right)_{e^0} = -\sum_{n=1}^{n_s}\dot{w}^{\text{rad}}_n e^0_n \tag{29}$$

Substituting each of these terms into the electron/electronic energy equation, Eq. (5), and simplifying leads to

$$\begin{aligned}\frac{\partial}{\partial t}(\rho e'') + \frac{\partial}{\partial\bar{x}}\cdot(\bar{V}\rho e'') &= -p_{e^-}\frac{\partial}{\partial\bar{x}}\cdot\bar{V} + \sum_{n=1}^{n_s}\xi_{e^-,n} + Q_{e^-} \\ &+ Q_{\text{elct}} + \dot{w}_{e^-}\frac{V^2}{2} + \sum_{n=1}^{n_s}\dot{w}^{\text{coll}}_n e_{e^0_n} - \frac{\partial}{\partial\bar{x}}\cdot\bar{q} \\ &+ \sum_{i=1}^{n_{\text{diss}}}\int_0^\infty \frac{Y_{\nu_i}(h\nu - \Delta E_{\text{elct}_i} - D_i)}{h\nu}\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu \\ &+ \sum_{i=1}^{n_{M\text{-band}}}\int_0^\infty \frac{Y_{\nu_i}(h\nu - E^u_{\text{elct}_i} + E^l_{\text{elct}_i})}{h\nu}\left(\frac{\partial}{\partial\bar{x}}\cdot\bar{q}_\nu\right)\mathrm{d}\nu\end{aligned} \tag{30}$$

where

$$e'' = \frac{\rho_{e^-}}{\rho}e_{e^-} + \sum_{n=1}^{n_s}\frac{\rho_n}{\rho}(e_{\text{elct}_n} + e_{e^0_n}) \tag{31}$$

The last three terms in Eq. (30) contain the effects of the absorption of radiation on the electron, electronic, and zero point energies of the gas. The first of these terms is the divergence of the radiative flux vector and contains the total energy absorbed or emitted by the gas. The last two terms in the equation correct this total energy by subtracting off the portion that does not affect the electron, electronic, or zero point energies. In essence, these terms subtract from the total energy absorbed or emitted the portion that affects only the heavy particle translational, vibrational, and rotational energies of the gas. The inclusion of these two terms in Eq. (30) and the zero point energy in Eq. (31) are where this formulation and those of Carlson,[1] Lee,[2] and Gnoffo[3] differ. The last two terms in Eq. (30) are expensive computationally and under most conditions will be small compared to the total radiative transfer; however, under certain conditions, such as in the radiation-dominated precursor, these terms can be important.[4]

The inclusion of the zero point energy in Eq. (31) is simply a matter of convenience. If it is not included then an additional term must be added to Eq. (30) to subtract this portion of the radiative energy. Also, the collisional mass production term must be removed from the right side of Eq. (30). In the bound-free radiative processes, most of the radiation absorbed or emitted affects the zero point energy of the gas through the production or destruction of ionized or dissociated species. If the quantity of radiation absorbed or emitted in the bound-free processes is relatively small, the zero point energy could be removed from Eq. (31) without adding the correction term to Eq. (30). However, as the bound-free processes become more significant, this could result in unrealistically large electron/electronic temperatures. In this case, the zero point energy should be included in Eq. (31). In order to properly include the influence of this term, however, it is also necessary to add the effects of radiative transfer on the chemical production of each species as given by Eq. (16).

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# Magnetized Plasma Flow over an Insulator at High Magnetic Reynolds Number

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## I. Introduction

ALTHOUGH magnetohydrodynamic (MHD) boundary layers have been the subject of considerable study,[1–5] little has been published on the physics of highly magnetized, high Reynolds number flows. The work presented here is an analysis of the self-similar solution to an idealized problem: the two-dimensional flow of a magnetized plasma moving above an insulating surface. This analysis can be used to understand the physics of magnetoplasmadynamic (MPD) thrusters such as the system considered by Di Capua and Jahn[6] and shown in Fig. 1.

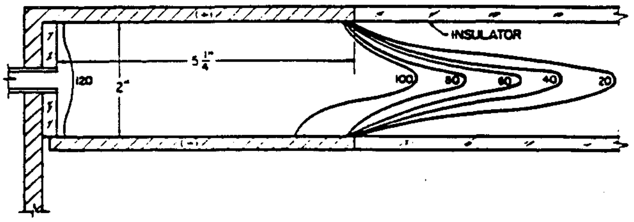

Fig. 1 Geometry of the experiment of Di Capua and Jahn.[6]

The definition of the problem is the following: The magnetic field is oriented in the $z$ direction. The plasma flows in the region $y > 0$. The boundary, located at $y = 0$, is a perfect conductor for $x < 0$ and an insulator for $x > 0$. At large distances from this boundary the plasma velocity is parallel to $y = 0$ and is oriented towards $x = +\infty$. The analysis is based on a simple one-fluid MHD model with constant transport coefficients. It is assumed that the Reynolds and the magnetic-Reynolds numbers are large. In addition to the classical viscous and thermal boundary layers, a magnetic boundary layer develops near the insulating surface. Strong electric currents appear in this boundary layer. Ohmic heating releases the electromagnetic energy convected by the plasma leading to an increase of its temperature.

The traditional approach to the analysis of compressible boundary layers is to find a similarity transformation derived from the Illingworth-Stewartson transformation, which turns the boundary-layer equations into a variation of the Blasius equation. A different approach is taken here. It consists of applying a simple geometric transformation to the coordinates and solving the resulting system of ordinary differential equations with a numerical solver. The results show that the conversion of the electromagnetic energy into thermal energy leads to high temperatures and heat transfer rates near the insulator surface, and that under certain conditions the boundary layer draws the flow towards the insulating surface.

## II. Formulation

In Cartesian coordinates the equations can be written

$$\frac{\partial}{\partial \bar{x}}(\bar{\rho}\bar{u}) + \frac{\partial}{\partial \bar{y}}(\bar{\rho}\bar{v}) = 0 \tag{1}$$

$$\begin{aligned}\bar{\rho}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{\rho}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial}{\partial \bar{x}}\left(\bar{P} + \frac{\bar{B}^2}{2\mu_0}\right) \\ = \frac{\partial}{\partial \bar{x}}\left(2\mu\frac{\partial \bar{u}}{\partial \bar{x}} + \lambda \operatorname{div} \bar{\boldsymbol{v}}\right) + \mu\frac{\partial}{\partial \bar{y}}\left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right)\end{aligned} \tag{2}$$

$$\begin{aligned}\bar{\rho}\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{\rho}\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial}{\partial \bar{y}}\left(\bar{P} + \frac{\bar{B}^2}{2\mu_0}\right) \\ = \mu\frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right) + \frac{\partial}{\partial \bar{y}}\left(2\mu\frac{\partial \bar{v}}{\partial \bar{y}} + \lambda \operatorname{div} \bar{\boldsymbol{v}}\right)\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{1}{\gamma - 1}\bar{\rho}^{\gamma}\left(\bar{u}\frac{\partial}{\partial \bar{x}} + \bar{v}\frac{\partial}{\partial \bar{y}}\right)\left(\frac{\bar{T}}{\bar{\rho}^{\gamma-1}}\right) = \frac{1}{\sigma\mu_0^2}\left[\left(\frac{\partial \bar{B}}{\partial \bar{x}}\right)^2\right. \\ \left. + \left(\frac{\partial \bar{B}}{\partial \bar{y}}\right)^2\right] + \lambda(\operatorname{div} \bar{\boldsymbol{v}})^2 + k\left(\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}\right) \\ + \mu\left[2\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 + 2\left(\frac{\partial \bar{v}}{\partial \bar{y}}\right)^2 + \left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right)^2\right]\end{aligned} \tag{4}$$

$$\bar{\rho}\bar{u}\frac{\partial}{\partial \bar{x}}\left(\frac{\bar{B}}{\bar{\rho}}\right) + \bar{\rho}\bar{v}\frac{\partial}{\partial \bar{y}}\left(\frac{\bar{B}}{\bar{\rho}}\right) = \frac{1}{\sigma\mu_0}\left(\frac{\partial^2 \bar{B}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{B}}{\partial \bar{y}^2}\right) \tag{5}$$

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